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Presymplectic representation of bi-Hamiltonian chains

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Abstract
Liouville integrable systems, which have bi-Hamiltonian representation of the Gel'fand–Zakharevich type, are considered. Bi-presymplectic representation of one-Casimir bi-Hamiltonian chains and weakly bi-presymplectic representation of multi-Casimir bi-Hamiltonian chains are constructed. The reduction procedure for Poisson and presymplectic structures is presented.

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1. Introduction

The bi-Poisson formulation of finite dimensional integrable Hamiltonian systems has been systematically developed over the last two decades (see [1] and the literature quoted therein). It has been found that most of the known Liouville integrable finite dimensional systems have more then one Hamiltonian representation. Moreover, in the majority of known cases, both Poisson structures of a given flow are degenerated. Perhaps this is the reason why such an important property of integrable systems was discovered so late, relative to the age of classical mechanics. For such systems, related bi-Poisson (bi-Hamiltonian) commuting vector fields belong to one or more bi-Hamiltonian chains starting and terminating with Casimirs of respective Poisson structures. An important aspect of such a construction is its relation to the recently developed geometric separability theory [2–10]. Actually, the necessary condition for the existence of separation coordinates is the reducibility of one of the Poisson structures onto a symplectic leaf of the other one. An important fact is that the whole procedure of variables separation is almost algorithmic.

On the other hand, it is well known from classical mechanics, that if the Poisson structure is nondegenerate, i.e. if the rank of the Poisson tensor is equal to the dimension of a phase space, then the phase space becomes a symplectic manifold with a symplectic structure being just the inverse of the Poisson structure. In such a case there exists an alternative (dual) description of Hamiltonian vector fields in the language of symplectic geometry. So, a natural question arises of whether one can construct such a dual picture in the degenerated case, when there is no natural inverse of the Poisson tensor [11].

A positive answer to this question is presented in the next sections of the paper. A dual presymplectic picture will be constructed for bi-Hamiltonian chains with one Casimir as well as with many Casimirs. The paper is organized as follows. In this section we recall some elementary facts from the Poisson and presymplectic geometry. In section 2 we introduce notions of dual pairs, compatible pairs and Poisson pairs and investigate some of their properties. In section 3, applying the results of the previous section, we construct a presymplectic representation of Poisson chains. In section 4 the deformation reduction procedure for Poisson and presymplectic chains is presented. Such a reduction is crucial for separability of underlying dynamical systems. Finally, in section 5, we illustrate the presented theory by a nontrivial example.

Given a manifold $\mathcal{M}$ of $\dim \mathcal{M} = m$, a *Poisson operator* Π of corank r on $\mathcal{M}$ is a bivector $\Pi \in \Lambda^2(\mathcal{M})$ with vanishing Schouten bracket:

$$[\Pi, \Pi]_S = 0, \tag{1}$$

whose kernel is spanned by exact 1-forms

$$\ker \Pi = Sp\{\mathrm{d}c_i\}_{i=1,\ldots,r}.$$

The symbol d denotes the operator of exterior derivative. In a given coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$ we have

$$\Pi = \sum_{i<j}^{m} \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

while the Poisson property (1) takes the form

$$\sum_l (\Pi^{lj}\partial_l \Pi^{ik} + \Pi^{il}\partial_l \Pi^{kj} + \Pi^{kl}\partial_l \Pi^{ji}) = 0, \qquad \partial_i := \frac{\partial}{\partial x^i}.$$

A function $c : \mathcal{M} \to \mathbb{R}$ is called the *Casimir function* of the Poisson operator Π if $\Pi\, \mathrm{d}c = 0$. A linear combination $\Pi_\lambda = \Pi_1 - \lambda \Pi_0$ ($\lambda \in \mathbb{R}$) of two Poisson operators Π_0 and Π_1 is called a *Poisson pencil* if the operator Π_λ is Poisson for any value of the parameter λ. In this case we say that Π_0 and Π_1 are *compatible*. A vector field X_F related to a function F through the relation

$$X_F = \Pi\, \mathrm{d}F \tag{2}$$

is called a Hamiltonian vector field with respect to the Poisson operator Π. It is also important to note that if X is any vector field on $\mathcal{M}$ that is Hamiltonian with respect to Π, then $L_X \Pi = 0$, where L_X is the Lie-derivative operator in the direction X.

Further, a *presymplectic operator* Ω on $\mathcal{M}$ defines a 2-form that is closed, i.e. $\mathrm{d}\Omega = 0$, degenerated in general. In the coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$ we can always represent Ω as

$$\Omega = \sum_{i<j}^{m} \Omega_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

where the closeness condition takes the form

$$\partial_i \Omega_{jk} + \partial_k \Omega_{ij} + \partial_j \Omega_{ki} = 0.$$

Moreover, the kernel of any presymplectic form is always an integrable distribution. A vector field X^F related to a function F by the relation

$$\Omega X^F = \mathrm{d}F \tag{3}$$

is called the inverse Hamiltonian vector field with respect to the presymplectic operator Ω. Generally, if Ω is a closed 2-form and X is an arbitrary vector field then

$$L_X\Omega = \mathrm{d}(\Omega X). \tag{4}$$

Hence, if $\Omega(Y) = 0$ for some vector field Y on $\mathcal{M}$ then $L_Y\Omega = 0$. Note that contrary to the Poisson case, a linear combination of two presymplectic operators is always presymplectic.

Poisson tensor Π, considered as the mapping $\Pi : T^*\mathcal{M} \to T\mathcal{M}$, induces a Lie bracket on the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on $\mathcal{M}$

$$\{\cdot,\cdot\}_\Pi : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \qquad \{F, G\}_\Pi \overset{\text{def}}{=} \langle \mathrm{d}F, \Pi\, \mathrm{d}G\rangle = \Pi(\mathrm{d}F, \mathrm{d}G), \tag{5}$$

(where $\langle\cdot,\cdot\rangle$ is the dual map between $T\mathcal{M}$ and $T^*\mathcal{M}$) which is skew-symmetric and satisfies Jacobi identity. It is called a *Poisson bracket*.

When a Poisson operator Π is nondegenerate, one can always define its inverse $\Omega = \Pi^{-1}$, called a *symplectic operator*, and then equations (2) and (3) are equivalent. Moreover, any Hamiltonian vector field with respect to Π is simultaneously the inverse Hamiltonian with respect to Ω and $X_F = X^F$. Finally, the symplectic operator Ω defines the same Poisson bracket as the related Poisson operator Π

$$\{F, G\}^\Omega := \Omega(X^F, X^G) = \langle \Omega X^F, X^G\rangle = \langle \mathrm{d}F, \Pi\, \mathrm{d}G\rangle = \{F, G\}_\Pi. \tag{6}$$

The equivalence is destroyed in the case of degeneracy. First, one cannot define Ω as the inverse of Π. Second, for degenerated Π equation (2) is valid for an arbitrary function F (as in the nondegenerate case), while for degenerated Ω and an arbitrary F there is no such vector field X^F that (3) is fulfilled. It means that equation (3) is valid only for a particular class of functions (contrary to the nondegenerate case). Finally it is not clear how to define a Poisson bracket with respect to a presymplectic form.

2. Dual Poisson–presymplectic pairs and compatible structures

In this section we introduce basic objects important for the theory, further develop and them investigate some of their properties. As the concept of dual pairs was introduced and developed for the first time in our previous paper [12], here we only recall their main properties. Let us remark that the concept of dual Poisson–presymplectic pairs [12], which we are going to apply to bi-Poisson chains, is a useful particular realization of the concept of Poisson brackets on presymplectic manifolds, presented by Dubrovin *et al* [11].

Consider a smooth manifold M of dimension m equipped with a pair of antisymmetric operators Π, Ω.

Definition 1. *A pair of antisymmetric tensor fields* (Π, Ω) *such that* $\Pi : T^*\mathcal{M} \to T\mathcal{M}$, *i.e.* Π *is twice contravariant, and* $\Omega : T\mathcal{M} \to T^*\mathcal{M}$, *i.e.* Ω *is twice covariant, is called a dual pair if there exist r 1-forms* $\alpha_i, i = 1, \ldots, r$, *and r linearly independent vector fields* $Z_i, i = 1, \ldots, r$, *such that the following conditions are satisfied:*

1. $\alpha_i(Z_j) = \delta_{ij}$ *for all* $i, j = 1, \ldots, r$.
2. *The kernel of* Π *is spanned by all* α_i, $\ker(\Pi) = Sp\{\alpha_i\}_{i=1,\ldots,r}$.
3. *The kernel of* Ω *is spanned by all the vector fields* Z_i, $\ker(\Omega) = Sp\{Z_i\}_{i=1,\ldots,r}$.
4. *The following partition of unity holds on* $T\mathcal{M}$

$$I = \Pi\Omega + \sum_{i=1}^{r} Z_i \otimes \alpha_i \tag{7}$$

where $\otimes$ *denotes the tensor product.*

Note that the partition of unity (7) on $T^*\mathcal{M}$ takes the form

$$I = \Omega\Pi + \sum_{i=1}^{r} \alpha_i \otimes Z_i. \tag{8}$$

Let us choose the basic 1-forms α_i in such a way that $\alpha_i = dc_i$ and let us denote a foliation of $\mathcal{M}$ given by the functions c_i by $\mathcal{N}$. This foliation consists of the leaves $\mathcal{N}_\nu = \{x \in M : c_i(x) = \nu_i, i = 1, \ldots, r\}, \nu = (\nu_r, \ldots, \nu_r)$. Condition 1 of the above definition implies that the distribution $\mathcal{Z}$ spanned by the vector fields Z_i is transversal to the foliation $\mathcal{N}$. Thus, for any $x \in \mathcal{M}$ we have

$$T_x\mathcal{M} = T_x\mathcal{N}_\nu \oplus \mathcal{Z}_x, \qquad T_x^*\mathcal{M} = T_x^*\mathcal{N}_\nu \oplus \mathcal{Z}_x^* \tag{9}$$

where $\mathcal{N}_\nu$ is a leaf from the foliation $\mathcal{N}$ that passes through x, the symbol $\oplus$ denotes the direct sum of the vector spaces, $\mathcal{Z}_x$ is the subspace of $T_x\mathcal{M}$ spanned by the vectors Z_i at this point, $T_x^*\mathcal{N}_\nu$ is the annihilator of $\mathcal{Z}_x$ and $\mathcal{Z}_x^*$ is the annihilator of $T_x\mathcal{N}_\nu$. Condition 2 of the above definition implies that $\mathrm{Im}(\Pi) = T\mathcal{N}$, condition 3 means that $\mathrm{Im}(\Omega) = T^*\mathcal{N}$ and condition 4 describes the degree of degeneracy of our pair.

Definition 2. *A dual pair* (Π, Ω) *is called a dual Poisson–presymplectic pair (in short: dual P–p pair) if* Π *is a Poisson bivector and if* Ω *is a closed 2-form.*

Note that in the case when a dual P–p pair has no degeneration ($r = 0$), we get the usual Poisson–symplectic pair of mutually inverse operators, since (7) reads then as $I = \Pi\Omega$. Moreover, for a degenerated case, when $r \neq 0$, as Ω is presymplectic, then $\ker(\Omega)$ is an integrable distribution with $[Z_i, Z_j] = 0, i, j = 1, \ldots, r$, and for Π Poisson, α_i are exact one-forms generated by Casimir functions: $\alpha_i = \mathrm{d}c_i, i = 1, \ldots, r$. The commutativity of Z_i follows from condition 1 of definition 1. The following lemma will be useful in further considerations.

Lemma 3. *Let* (Π, Ω) *be a dual P–p pair, then*

$$L_{Z_i}\Pi = 0, \qquad i = 1, \ldots, r.$$

Assume that (Π, Ω) is a dual P–p pair and

$$\Pi\,\mathrm{d}F = X_F \tag{10}$$

is a Hamiltonian vector field with respect to Π. Applying Ω to both sides of (10) and using the decomposition (8) we get

$$\mathrm{d}F = \Omega(X_F) + \sum_{i=1}^{r} Z_i(F)\,\mathrm{d}c_i, \tag{11}$$

which reconstructs $\mathrm{d}F$ from X_F and $Z_i(F)$ in the case of degenerated Poisson structure Π. In that sense Ω plays the role of the 'inverse' of Π. Note that inverse Hamiltonian vector fields with respect to Ω are related to functions which are annihilated by $\ker(\Omega)$, i.e. $Z_i(F) = 0, i = 1, \ldots, r$. Then, equation (11) reduces to (3) with $\Omega(X_F) = \Omega(X^F)$. It means that X_F is not only a Hamiltonian but also inverse Hamiltonian vector field related to the same Hamiltonian function F. Moreover, it is a gauge freedom for inverse Hamiltonian vector fields X^F with respect to Ω. Indeed, applying Π to both sides of equation (3) and using decomposition (7) one gets

$$X^F - X_F = \sum_i X^F(c_i) Z_i.$$

It means that an inverse Hamiltonian vector field X^F is simultaneously a Hamiltonian vector field, i.e. $X^F = X_F$, if X^F annihilates the kernel of Π.

The definition of dual objects is not unique and questions about the 'gauge freedom' can be posed. A possible realization of such a freedom is as follows: given a dual P–p pair (Π, Ω) we are looking for possible deformations of Ω to get a new presymplectic form Ω' ensuring that (Π, Ω') is dual again. Another possibility is related to a gauge freedom for the operator Π, i.e. how can we deform Π to a new Poisson bivector Π' so that (Π', Ω) is also the dual pair. An example of such a gauge freedom is given in the following proposition:

Proposition 4. *Let (Π, Ω) be a dual P–p pair as in definitions 1 and 2. Suppose that F_i are real functions on $\mathcal{M}$ related to vector fields K_i which are simultaneously Hamiltonian and inverse Hamiltonian with respect to (Π, Ω) pair*

$$\mathrm{d}F_i = \Omega K_i, \qquad K_i = \Pi\,\mathrm{d}F_i, \qquad i = 1, \ldots, r.$$

Then

(i)

$$\Omega' = \Omega + \sum_i \mathrm{d}F_i \wedge \mathrm{d}c_i,$$

is a dual to Π presymplectic 2-form, provided that

$$\Pi(\mathrm{d}F_i, \mathrm{d}F_j) = 0 \qquad \textit{for all } i, j.$$

(ii)

$$\Pi' = \Pi + \sum_i Z_i \wedge K_i$$

is a dual to the Ω Poisson bivector, provided that

$$\Omega(K_i, K_j) = 0 \qquad \textit{for all } i, j.$$

Let us now turn our attention to brackets induced on the space $C^\infty(\mathcal{M})$. We know that the Poisson operator Π turns $C^\infty(\mathcal{M})$ into a Poisson algebra with the Poisson bracket (5)

$$\{F, G\}_\Pi = \Pi(\mathrm{d}F, \mathrm{d}G) = \langle \mathrm{d}F, \Pi\,\mathrm{d}G\rangle.$$

In case when Ω is a part of a dual P–p pair we can define the above bracket through the Ω in the following way:

Lemma 5. *Let (Π, Ω) be a dual P–p pair. Define a new bracket on $C^\infty(\mathcal{M})$*

$$\{F, G\}^\Omega := \Omega(X_F, X_G) = \langle \Omega X_F, X_G\rangle, \qquad X_F = \Pi\,\mathrm{d}F.$$

Then $\{\cdot,\cdot\}^\Omega = \{\cdot,\cdot\}_\Pi$, i.e. both brackets are identical.

The proofs of lemma 3, lemma 5 and proposition 4, as well as more details on the concept of dual P–p pairs the reader can find in [12].

Now we pass to the concept of compatibility.

Definition 6. *A Poisson bivector Π and presymplectic two-form Ω are called a compatible P–p pair if $\Omega_D := \Omega\Pi\Omega$ is presymplectic.*

As well known (see for example [1]) if (Π, Ω) is a compatible P–p pair, then the second order tensor $\Phi = \Pi\Omega : T\mathcal{M} \to T\mathcal{M}$ has vanishing Nijenhuis torsion

$$L_{\Phi\tau}\Phi - \Phi L_\tau \Phi = 0, \qquad \forall \tau \in T\mathcal{M},$$

and is called a hereditary operator or recursion operator. Moreover, $\Pi_D := \Pi\Omega\Pi$ is a Poisson bivector. Observe that a dual P–p pair (Π_0, Ω_0) is a trivial example of a compatible pair as

$$\Omega_D = \Omega_0\Pi_0\Omega_0 = \Omega_0\left(I - \sum_i Z_i \otimes \mathrm{d}c_i\right) = \Omega_0. \tag{12}$$

Lemma 7. *If Ω is a presymplectic 2-form compatible with a Poisson bivector Π_0, then the bracket*

$$\{F, G\}^{\Omega} := \Omega\big(X_F^0, X_G^0\big), \qquad X_F^0 = \Pi_0\,\mathrm{d}F$$

is a Poisson bracket.

Proof.

$$\begin{aligned}\{F, G\}^{\Omega} &= \big\langle \Omega X_F^0, X_G^0\big\rangle = \langle \Omega\Pi_0\,\mathrm{d}F, \Pi_0\,\mathrm{d}G\rangle = -\langle \mathrm{d}G, \Pi_0\Omega\Pi_0\,\mathrm{d}F\rangle \\ &= \langle \mathrm{d}F, \Pi_0\Omega\Pi_0\,\mathrm{d}G\rangle = \langle \mathrm{d}F, \Pi_D\,\mathrm{d}G\rangle \\ &= \{F, G\}_{\Pi_D}\end{aligned}$$

and Π_D is Poisson. □

Obviously, when $\Omega = \Omega_0$, i.e. the compatible pair is simply a dual pair, then we deal with a special case described by lemma 5. Moreover, if (Π, Ω_0) is a compatible P–p pair and $\ker(\Omega_0) = Sp\{Z_i\}_{i=1,\ldots,r}$, then

$$\Omega_0\big(L_{Z_i}\Pi\big)\Omega_0 = 0, \qquad i = 1, \ldots, r, \tag{13}$$

which follows from (4).

Theorem 8. *Let (Π_0, Ω_0) be a dual P–p pair, such that* $\ker\Omega_0 = Sp\{Z_i\}$ *and* $\ker\Pi_0 = Sp\{\mathrm{d}c_i\}$*. Moreover, let Π be a Poisson bivector compatible with Ω_0, then*

(i)

$$\begin{aligned}\Pi_d &:= \Pi_0\Omega_D\Pi_0 = \Pi_0\Omega_0\Pi\Omega_0\Pi_0 \\ &= \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2}\sum_{i,j} c_{ij} Z_i \wedge Z_j,\end{aligned} \tag{14}$$

(ii)

$$L_{Z_i}\Pi_d = 0, \qquad i = 1, \ldots, r, \tag{15}$$

(iii)

$$L_{Z_l}\Pi = \sum_i [Z_l, X_i] \wedge Z_i - \frac{1}{2}\sum_{i,j} Z_l(c_{ij}) Z_i \wedge Z_j, \tag{16}$$

where $X_i = \Pi\,\mathrm{d}c_i$, $c_{ij} = \Pi(\mathrm{d}c_i, \mathrm{d}c_j) = \langle \mathrm{d}c_i, \Pi\,\mathrm{d}c_j\rangle$,

(iv) Π_d is Poisson.

Proof. From the definition of Π_d we have

$$\begin{aligned}\Pi_d = \Pi_0\Omega_0\Pi\Omega_0\Pi_0 &= \left(I - \sum_i Z_i \otimes \mathrm{d}c_i\right)\Pi\left(I - \sum_j \mathrm{d}c_j \otimes Z_j\right) \\ &= \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2}\sum_{i,j} c_{ij} Z_i \wedge Z_j.\end{aligned}$$

Then, from lemma 3 and relation (13), it follows that $L_{Z_i}\Pi_d = 0$. Next, from (i) and (ii) immediately follows (iii). Finally we prove the property (iv). If X, Y are some vector fields, then their Schouten bracket $[X, Y]_S = [X, Y] = L_X Y$ is a usual Lie bracket (commutator). Moreover, for arbitrary bivector P and function F, the Schouten bracket fulfils the relations

$$[X \wedge Y, P]_S = Y \wedge [X, P]_S - X \wedge [Y, P]_S, \qquad [X, P]_S = L_X P \tag{17}$$

and

$$L_{FX} P = F L_X P - (P\mathrm{d}F) \wedge X. \tag{18}$$

Now, using (17) and (18), after straightforward but lengthy calculations, one finds

$$\begin{aligned}
[\Pi_d, \Pi_d]_S = [\Pi, \Pi]_S &- 2\left[\Pi, \sum_i X_i \wedge Z_i\right]_S + \left[\Pi, \sum_{i,j} c_{ij} Z_i \wedge Z_j\right]_S \\
&+ \left[\sum_i X_i \wedge Z_i, \sum_j X_j \wedge Z_j\right]_S - \left[\sum_k X_k \wedge Z_k, \sum_{i,j} c_{ij} Z_i \wedge Z_j\right]_S \\
&+ \frac{1}{4}\left[\sum_{i,j} c_{ij} Z_i \wedge Z_j, \sum_{k,l} c_{kl} Z_k \wedge Z_l\right]_S \\
&= \sum_{i,j,k} X_k(c_{ij}) Z_i \wedge Z_k \wedge Z_j = 0,
\end{aligned}$$

as

$$\sum_{i,j,k} X_k(c_{ij}) Z_i \wedge Z_j \wedge Z_k = \frac{1}{3}\sum_{i,j,k}[X_k(c_{ij}) + X_k(c_{ij}) + X_k(c_{ij})] Z_i \wedge Z_k \wedge Z_j = 0$$

which follows from Jacobi identity. □

As the concept of compatibility will be important in the reduction scheme for bi-Hamiltonian chains, the following theorem will be useful in the further considerations.

Theorem 9. *Let* (Π_0, Ω_0) *be a dual P–p pair such that* $\ker \Omega_0 = Sp\{Z_i\}$ *and* Π *be a Poisson tensor compatible with* Π_0. *Then,* Π *is compatible with* Ω_0 *if*

$$\Omega_0(L_{Z_i}\Pi)\Omega_0 = 0, \qquad i = 1, \ldots, k. \tag{19}$$

Proof. First we gather all necessary formulae important for the calculation. For any Poisson operator Π

$$L_{\Pi\gamma}\Pi = -\Pi(\mathrm{d}\gamma)\Pi, \qquad \forall \gamma \in T^*M, \tag{20}$$

for any presymplectic form Ω

$$L_X\Omega = \mathrm{d}(\Omega X), \qquad \forall X \in TM \tag{21}$$

and for an arbitrary second-order mixed rank tensor Φ

$$[\Phi X_1, X_2] = \Phi[X_1, X_2] + \left(L_{X_2}\Phi\right)X_1. \tag{22}$$

For arbitrary vectors X_1, X_2, X 1-forms α_1, α_2, 2-form Ω and function F, the following relations hold:

$$\begin{aligned}
&(X_1 \otimes X_2)(\alpha_1 \otimes \alpha_2) = \alpha_1(X_2) X_1 \otimes \alpha_2, && \alpha_1(X_2) = \langle \alpha_1, X_2\rangle, \\
&\Pi(\alpha_1 \otimes \alpha_2) = \Pi(\alpha_1) \otimes \alpha_2, && \Omega(X_1 \otimes X_2) = \Omega(X_1) \otimes X_2, \\
&(\alpha_1 \otimes \alpha_2)\Pi = -\alpha_1 \otimes (\Pi\alpha_2), && (X_1 \otimes X_2)\Omega = -X_1 \otimes (\Omega X_2), \\
&L_{FX}\Omega = F L_X \Omega + \mathrm{d}F \wedge \Omega X.
\end{aligned} \tag{23}$$

As Π_0 and Π are compatible so $\Pi + \lambda\Pi_0$ is Poisson, hence for $\forall\tau \in TM$ and $\gamma = \Omega_0\tau$ from (20) we have

$$\begin{aligned} 0 &= L_{(\Pi+\lambda\Pi_0)\gamma}(\Pi + \lambda\Pi_0) + (\Pi + \lambda\Pi_0)\,\mathrm{d}\gamma(\Pi + \lambda\Pi_0) \\ &= \lambda\big(L_{\Pi\gamma}\Pi_0 + L_{\Pi_0\gamma}\Pi + \Pi(\mathrm{d}\gamma)\Pi_0 + \Pi_0(\mathrm{d}\gamma)\Pi\big). \end{aligned}$$

Applying (7), (20) and (18) we find

$$L_{\Pi\gamma}\Pi_0 = -\Pi_0\big(L_{\Pi\Omega_0\tau}\Omega_0\big)\Pi_0 - \sum_i \big(\Pi_0\,\mathrm{d}a^i_\gamma\big) \wedge Z_i,$$

where $a^i_\gamma = \langle \mathrm{d}c_i, \Pi\gamma\rangle$, $L_{Z_i}\Omega_0 = 0$ and

$$L_{\Pi_0\gamma}\Pi = L_\tau\Pi - \sum_i L_{\tau(c_i)Z_i}\Pi,$$

hence

$$0 = -\Pi_0\big(L_{\Pi\Omega_0\tau}\Omega_0\big)\Pi_0 + \sum_i L_{a^i_\gamma Z_i}\Pi_0 + L_\tau\Pi - \sum_i L_{\tau(c_i)Z_i}\Pi + \Pi(L_\tau\Omega_0)\Pi_0 + \Pi_0(L_\tau\Omega_0)\Pi.$$

Multiplying from left and right by Ω_0 and using (7), after strenuous but straightforward calculations with the application of formulae (20)–(23) we arrive at the relation

$$0 = -\mathrm{d}(\Omega_0\Pi\Omega_0\tau) + L_\tau(\Omega_0\Pi\Omega_0) - \sum_i \big[\Omega_0\big(L_{Z_i}\Pi\big)\Omega_0\big]\tau \wedge \mathrm{d}c_i - \sum_i \tau(c_i)\Omega_0\big(L_{Z_i}\Pi\big)\Omega_0.$$

Hence, $\Omega_0\Pi\Omega_0$ is closed if

$$\sum_i \big[\Omega_0\big(L_{Z_i}\Pi\big)\Omega_0\big]\tau \wedge \mathrm{d}c_i + \sum_i \tau(c_i)\Omega_0\big(L_{Z_i}\Pi\big)\Omega_0 = 0.$$

As the last equality holds for an arbitrary vector field τ, hence

$$\Omega_0\big(L_{Z_i}\Pi\big)\Omega_0 = 0, \qquad i = 1, \ldots, r.$$

□

Definition 10. *Let* (Π_0, Ω_0) *be a dual P–p pair and* Π *be a Poisson bivector. We say that* Π *is compatible with the pair* (Π_0, Ω_0) *if* Π *is compatible with* Π_0 *and* Ω_0.

Up to now, we have induced a Poisson bracket on $C^\infty(\mathcal{M})$ in various ways using not only Poisson bivectors but also dual pairs and compatible pairs. So, the question is what is the most general way of introducing a Poisson algebra on $C^\infty(\mathcal{M})$.

Definition 11. *Assume that* Π *is some bivector and* Ω *is a 2-form. A pair* (Π, Ω) *is called a Poisson pair if* $\Pi_D = \Pi\Omega\Pi$ *is Poisson. Two Poisson pairs* (Π_1, Ω_1) *and* (Π_2, Ω_2) *will be called equivalent if* $\Pi_1\Omega_1\Pi_1 = \Pi_2\Omega_2\Pi_2$.

Each compatible pair is simultaneously a Poisson pair. For a given Poisson pair (Π, Ω) the bracket

$$\begin{aligned} \{F, G\}^\Omega_\Pi &:= \Omega(\Pi\,\mathrm{d}F, \Pi\,\mathrm{d}G) = \langle \Omega\Pi\,\mathrm{d}F, \Pi\,\mathrm{d}G\rangle = \langle \mathrm{d}F, \Pi\Omega\Pi\,\mathrm{d}G\rangle \\ &= (\Pi\Omega\Pi)(\mathrm{d}F, \mathrm{d}G) = \{F, G\}_{\Pi_D} \end{aligned}$$

is a Poisson bracket. Hence, the property of closeness of Ω is too strong for the definition of a Poisson algebra.

Definition 12. *Let* Π *be a bivector with a kernel spanned by exact 1-forms. A 2-form* Ω *is called weakly presymplectic with respect to* Π *if it is closed on* $\mathrm{Im}\Pi = T\mathcal{N}$*, where* $\mathcal{N}$ *is the foliation given by functions whose differentials span the kernel of* Π.

Obviously, if (Π, Ω) is a Poisson pair then Ω is weakly presymplectic with respect to Π. As we will see later, weakly presymplectic forms play an important role in bi-Hamiltonian chains and in the reduction procedure.

3. Presymplectic representation of Gel'fand–Zakharevich chains

Let us consider a bi-Poisson manifold (M, Π_0, Π_1) of $\dim M = m = 2n + r$ where Π_0, Π_1 is a pair of compatible Poisson tensors of rank $2n$. Moreover, we assume that the Poisson pencil Π_λ admits r, polynomial with respect to the pencil parameter λ, Casimir functions of the form

$$H^{(j)}(\lambda) = \sum_{i=0}^{n_j} H_i^{(j)} \lambda^{n_j - i}, \qquad j = 1, \ldots, r, \tag{24}$$

such that $n_1 + \cdots + n_r = n$ and $H_i^{(j)}$ are functionally independent. The collection of n bi-Hamiltonian vector fields

$$X_i^{(j)} = \Pi_1 \, \mathrm{d}H_{i-1}^{(j)} = \Pi_0 \, \mathrm{d}H_i^{(j)}, \qquad i = 1, \ldots, n_j, \quad j = 1, \ldots, r, \tag{25}$$

constructed from Casimirs of the pencil

$$\Pi_\lambda \, \mathrm{d}H^{(j)}(\lambda) = 0,$$

is called the Gel'fand–Zakharevich system of the bi-Poisson manifold $\mathcal{M}$ [13, 14]. Note that each chain starts from a Casimir of Π_0 and terminates with a Casimir of Π_1. Moreover all $H_i^{(j)}$ pairwise commute with respect to both Poisson structures

$$X_i^{(j)}\big(H_l^{(k)}\big) = \big\langle \mathrm{d}H_l^{(k)}, \Pi_0 \, \mathrm{d}H_i^{(j)} \big\rangle = \big\langle \mathrm{d}H_l^{(k)}, \Pi_1 \, \mathrm{d}H_{i-1}^{(j)} \big\rangle = 0.$$

$$\Updownarrow$$

$$\Pi_\lambda\big(\mathrm{d}H_i^{(j)}, \mathrm{d}H_l^{(k)}\big) = 0.$$

3.1. Bi-presymplectic representation of one-Casimir chains

As in this subsection we restrict our considerations to the simplest case of $r = 1$, i.e. to the one-Casimir case, we will use the following notation for a single bi-Hamiltonian chain

$$X_i = \Pi_0 \, \mathrm{d}H_i = \Pi_1 \, \mathrm{d}H_{i-1}, \qquad i = 0, \ldots, n+1. \tag{26}$$

The chain starts with a Casimir H_0 of Π_0 and terminates with a Casimir H_n of Π_1.

Let Ω_0 be a dual to Π_0 presymplectic form. The kernels of Ω_0 and Π_0 are one dimensional: $\ker \Omega_0 = Z$, $\ker \Pi_0 = \mathrm{d}H_0$ and

$$L_Z \Omega_0 = 0, \qquad L_Z \Pi_0 = 0.$$

We assume that $\Omega_0 (L_Z \Pi_1) \Omega_0 = 0$, i.e. that Π_1 is compatible with the P–p pair (Π_0, Ω_0), so

$$L_Z \Pi_1 = [Z, X_1] \wedge Z, \qquad X_1 = \Pi_1 \, \mathrm{d}H_0$$

and

$$\Omega_{1D} := \Omega_0 \Pi_1 \Omega_0$$

is also presymplectic with $\ker \Omega_0 \subseteq \ker \Omega_{1D}$.

Next, we construct the following 2-form:

$$\Omega_1 = \Omega_{1D} + \Omega_0 X_1 \wedge \mathrm{d}H_0 = \Omega_{1D} + \mathrm{d}H_1 \wedge \mathrm{d}H_0.$$

It is obviously a presymplectic form. Moreover, (Π_0, Ω_1) is a Poisson pair. Indeed,

$$\Pi_0 \Omega_1 \Pi_0 = \Pi_0 \Omega_{1D} \Pi_0 + \Pi_0 (\mathrm{d}H_1 \wedge \mathrm{d}H_0) \Pi_0 = \Pi_0 \Omega_{1D} \Pi_0 = \Pi_{1d} = \Pi_1 - X_1 \wedge Z \tag{27}$$

which is Poisson according to theorem 8.

Lemma 13. *Vector field $Y = X_n + Z(H_n) Z$ belongs to* $\ker \Omega_1$.

Proof.

$$\begin{aligned}\Omega_1 Y &= (\Omega_{1D} - \mathrm{d}H_0 \wedge \mathrm{d}H_1)(X_n + Z(H_n)Z)\\ &= (\Omega_0\Pi_1\Omega_0)X_n - Z(H_n)Z(H_1)\,\mathrm{d}H_0 + Z(H_n)\,\mathrm{d}H_1.\end{aligned}$$

On the other hand, from (11) and the fact that H_0 is the only Casimir function of Π_0

$$\begin{aligned}(\Omega_0\Pi_1\Omega_0)X_n &= \Omega_0\Pi_1(\mathrm{d}H_n - Z(H_n)\,\mathrm{d}H_0) = -Z(H_n)\Omega_0 X_1\\ &= -Z(H_n)(\mathrm{d}H_1 - Z(H_1)\,\mathrm{d}H_0)\\ &= -Z(H_n)\,\mathrm{d}H_1 + Z(H_n)Z(H_1)\,\mathrm{d}H_0.\end{aligned}$$

□

Now we are prepared to formulate the following theorem:

Theorem 14. *Bi-presymplectic representation of the bi-Poisson chain (26) takes the form*

$$\beta_i = \Omega_0 Y_i = \Omega_1 Y_{i-1}, \qquad i = 0, \ldots, n+1, \tag{28}$$

where

$$Y_i = X_i + Z(H_i)Z, \qquad \beta_i = \mathrm{d}H_i - Z(H_i)\,\mathrm{d}H_0.$$

The chain starts with a kernel vector field $Y_0 = Z$ of Ω_0 and terminates with a kernel vector field $Y_n \equiv Y = X_n + Z(H_n)Z$ of Ω_1.

Proof.

$$\begin{aligned}\Omega_0 Y_i &= \Omega_0 X_i,\\ \Omega_1 Y_{i-1} &= (\Omega_0\Pi_1\Omega_0 - \mathrm{d}H_0 \wedge \mathrm{d}H_1)(X_{i-1} + Z(H_{i-1})Z)\\ &= (\Omega_0\Pi_1\Omega_0)X_{i-1} - Z(H_{i-1})Z(H_1)\,\mathrm{d}H_0 + Z(H_{i-1})\,\mathrm{d}H_1,\end{aligned}$$

$$\begin{aligned}(\Omega_0\Pi_1\Omega_0)X_{i-1} &= \Omega_0\Pi_1(\mathrm{d}H_{i-1} - Z(H_{i-1})\,\mathrm{d}H_0) = \Omega_0(X_i) - Z(H_{i-1})X_1\\ &= \Omega_0 X_i - Z(H_{i-1})\Omega_0 X_1\\ &= \Omega_0 X_i - Z(H_{i-1})\,\mathrm{d}H_1 + Z(H_{i-1})Z(H_1)\,\mathrm{d}H_0.\end{aligned}$$

□

Observe that neither X_i nor Y_i vector fields are inverse Hamiltonian with respect to Ω_0 and Ω_1. Besides $[Y_i, Y_j] \neq 0$. Introducing a presymplectic pencil

$$\Omega_\lambda = \Omega_1 - \lambda\Omega_0$$

with a kernel vector field

$$Y = \sum_{i=0}^{n} Y_i \lambda^{n-i},$$

the bi-presymplectic chain (28) takes the form $\Omega_\lambda Y = 0$. On the other hand, the pairs (Π_0, Ω_0) and (Π_0, Ω_1) are Poisson pairs, hence Ω_0 and Ω_1 define Poisson brackets. The first one is equal to that given by Π_0 (12) while the second one is equal to that given by Π_{1d} (27). Moreover,

$$\Omega_0(X_i, X_j) = \{H_i, H_j\}_{\Pi_0} = 0, \qquad \Omega_1(X_i, X_j) = \{H_i, H_j\}_{\Pi_{1d}} = 0.$$

The first bracket is obvious, the second one follows from the relation

$$\begin{aligned}\Omega_1 X_i &= (\Omega_{1D} + \mathrm{d}H_1 \wedge \mathrm{d}H_0)X_i = \Omega_0\Pi_1\Omega_0 X_i = \Omega_0\Pi_1(\mathrm{d}H_i) - Z(H_i)\,\mathrm{d}H_0\\ &= \Omega_0 X_{i+1} - Z(H_i)\Omega_0 X_1\end{aligned}$$

and the first bracket. Additionally, Poisson tensors Π_0 and Π_{1d} are compatible as

$$[\Pi_{1d}, \Pi_0]_S = [\Pi_1 - X_1 \wedge Z, \Pi_0]_S = X_1 \wedge [Z, \Pi_0]_S - Z \wedge [X_1, \Pi_0]_S = 0.$$

As a consequence (Π_0, Ω_λ) is a Poisson pair and

$$\Omega_\lambda(X_i, X_j) = 0.$$

3.2. Weakly bi-presymplectic representation of multi-Casimir chains

In this subsection we will show that bi-presymplectic representation is purely an one-Casimir phenomenon. Consider the r-Casimir Gel'fand–Zakharevich chain (24), (25). Let Ω_0 be a dual to Π_0 presymplectic form. The kernels of Ω_0 and Π_0 are r-dimensional: $\ker\Omega_0 = Sp\{Z_i\}_{i=1,\ldots,r}$, $\ker\Pi_0 = Sp\{\mathrm{d}H_0^{(i)}\}_{i=1,\ldots,r}$ and

$$L_{Z_i}\Omega_0 = 0, \qquad L_{Z_i}\Pi_0 = 0, \qquad i = 1,\ldots,r. \tag{29}$$

We assume that $\Omega_0\big(L_{Z_i}\Pi_1\big)\Omega_0 = 0$, i.e. that Π_1 is compatible with the P–p pair (Π_0, Ω_0), so from involutivity of $H_k^{(i)}$ relation (16) takes the form

$$L_{Z_i}\Pi_1 = \sum_k \big[Z_i, X_1^{(k)}\big] \wedge Z_k, \qquad X_1^{(k)} = \Pi_1\,\mathrm{d}H_0^{(k)}$$

and

$$\Omega_{1D} := \Omega_0\Pi_1\Omega_0$$

is also presymplectic with $\ker\Omega_0 \subseteq \ker\Omega_{1D}$.

Next, we construct the following 2-forms

$$\overline{\Omega}_1 = \Omega_{1D} + \sum_{j=1}^{r} \Omega_0 X_1^{(j)} \wedge \mathrm{d}H_0^{(j)}, \qquad \Omega_1 = \Omega_{1D} + \sum_{j=1}^{r} \mathrm{d}H_1^{(j)} \wedge \mathrm{d}H_0^{(j)},$$

related with each other as follows

$$\Omega_1 = \overline{\Omega}_1 + \frac{1}{2}\sum_{k,l} A_{kl}\,\mathrm{d}H_0^{(k)} \wedge \mathrm{d}H_0^{(l)}, \qquad A_{kl} = Z_k\big(H_1^{(l)}\big) - Z_l\big(H_1^{(k)}\big).$$

Obviously Ω_1 is presymplectic and together with Π_0 forms a Poisson pair as

$$\Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_{1D}\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0 = \Pi_{1d} = \Pi_1 - \sum_i X_1^{(i)} \wedge Z_i$$

is Poisson. It is also clear that $\overline{\Omega}_1$ is not closed as

$$\mathrm{d}\overline{\Omega}_1 = -\frac{1}{2}\sum_{k,l} \mathrm{d}A_{kl} \wedge \mathrm{d}H_0^{(k)} \wedge \mathrm{d}H_0^{(l)},$$

but is weakly presymplectic with respect to Π_0

$$\mathrm{d}\overline{\Omega}_1(\Pi_0\alpha_1, \Pi_0\alpha_2, \Pi_0\alpha_3) = 0, \qquad \forall \alpha_1, \alpha_2, \alpha_3 \in T^*\mathcal{M}.$$

Moreover, $(\Pi_0, \overline{\Omega}_1)$ is a Poisson pair equivalent to the (Π_0, Ω_1) one as $\Pi_0\overline{\Omega}_1\Pi_0 = \Pi_0\Omega_1\Pi_0 = \Pi_{1d}$.

Theorem 15. *Multi-Casimir bi-Poisson chains (25) have weakly bi-presymplectic representation*

$$\beta_i^{(j)} = \Omega_0 Y_i^{(j)} = \overline{\Omega}_1 Y_{i-1}^{(j)}, \qquad j = 1,\ldots,r, \qquad i = 0,\ldots,n_j+1, \tag{30}$$

where

$$Y_i^{(j)} = X_i^{(j)} + \sum_{k=1}^{r} Z_k\big(H_i^{(j)}\big) Z_k, \qquad \beta_i^{(j)} = \mathrm{d}H_i^{(j)} - \sum_{k=1}^{r} Z_k\big(H_i^{(j)}\big)\,\mathrm{d}H_0^{(k)}.$$

The jth chain starts with a kernel vector field $Y_0^{(j)} = Z_j$ of Ω_0 and terminates with a kernel vector field $Y_{n_j}^{(j)} = X_{n_j}^{(j)} + \sum_{k=1}^{m} Z_k\big(H_{n_j}^{(j)}\big)Z_k$ of $\overline{\Omega}_1$.

Proof. We have

$$\Omega_0 Y_i^{(j)} = \Omega_0 X_i^{(j)}.$$

On the other hand,

$$\begin{aligned}\overline{\Omega}_1 Y_{i-1}^{(j)} &= \left(\Omega_0\Pi_1\Omega_0 + \sum_l \Omega_0 X_1^{(l)} \wedge \mathrm{d}H_0^{(l)}\right)\left(X_{i-1}^{(j)} + \sum_k Z_k\big(H_{i-1}^{(j)}\big)Z_k\right)\\ &= \Omega_0\Pi_1\Omega_0 X_{i-1}^{(j)} + \left(\sum_l \Omega_0 X_1^{(l)} \wedge \mathrm{d}H_0^{(l)}\right) X_{i-1}^{(j)}\\ &\quad + \sum_{l,k} Z_k\big(H_{i-1}^{(j)}\big)\big(\Omega_0 X_1^{(l)} \wedge \mathrm{d}H_0^{(l)}\big)Z_k.\end{aligned}$$

Using decomposition (11) and bi-Hamiltonian chains (25) one finds

$$\Omega_0\Pi_1\Omega_0 X_{i-1}^{(j)} = \Omega_0 X_i^{(j)} - \sum_k Z_k\big(H_{i-1}^{(j)}\big)\,\mathrm{d}H_1^{(k)} + \sum_{l,k} Z_k\big(H_{i-1}^{(j)}\big)Z_l\big(H_1^{(k)}\big)\,\mathrm{d}H_0^{(l)},$$

$$\sum_{l,k} Z_k\big(H_{i-1}^{(j)}\big)\big(\Omega_0 X_1^{(l)} \wedge \mathrm{d}H_0^{(l)}\big)Z_k = \sum_k Z_k\big(H_{i-1}^{(j)}\big)\,\mathrm{d}H_1^{(k)} - \sum_{j,k} Z_k\big(H_{i-1}^{(j)}\big)Z_l\big(H_1^{(k)}\big)\,\mathrm{d}H_0^{(l)},$$

$$\left(\sum_l \Omega_0 X_1^{(l)} \wedge \mathrm{d}H_0^{(l)}\right) X_{i-1}^{(j)} = -\sum_l \Omega_0\big(X_1^{(l)}, X_{i-1}^{(j)}\big)\,\mathrm{d}H_0^{(l)} = 0.$$

The last equality follows from the fact that $\Omega_0\big(X_1^{(l)}, X_{i-1}^{(j)}\big) = \Pi_0\big(\mathrm{d}H_1^{(l)}, \mathrm{d}H_{i-1}^{(j)}\big) = 0$. Hence

$$\overline{\Omega}_1 Y_{i-1}^{(j)} = \Omega_0 X_i^{(j)}.$$

□

Introducing a weakly presymplectic pencil

$$\overline{\Omega}_\lambda = \overline{\Omega}_1 - \lambda\Omega_0$$

with respect to Π_0, with kernel vector fields

$$Y^{(j)} = \sum_{i=0}^{n_j} Y_i^{(j)}\lambda^{n_j - i}, \qquad j = 1, \ldots, r,$$

the weakly bi-presymplectic chains (30) take the form $\overline{\Omega}_\lambda Y^{(j)} = 0$. On the other hand, as we mentioned before, the pairs (Π_0, Ω_0) and $(\Pi_0, \overline{\Omega}_1)$ are Poisson pairs, hence Ω_0 and $\overline{\Omega}_1$ define Poisson brackets. The first one is equal to that given by Π_0 while the second one is equal to that given by Π_{1d}. Moreover,

$$\Omega_0\big(X_i^{(k)}, X_j^{(l)}\big) = \big\{H_i^{(k)}, H_j^{(l)}\big\}_{\Pi_0} = 0, \qquad \Omega_1\big(X_i^{(k)}, X_j^{(l)}\big) = \big\{H_i^{(k)}, H_j^{(l)}\big\}_{\Pi_{1d}} = 0.$$

The first bracket is obvious; the second one follows from the relation

$$\begin{aligned}\Omega_1 X_i^{(k)} &= \left(\Omega_{1D} + \sum_r \mathrm{d}H_1^{(r)} \wedge \mathrm{d}H_0^{(r)}\right) X_i^{(k)} = \Omega_0\Pi_1\Omega_0 X_i^{(k)}\\ &= \Omega_0\Pi_1\left(\mathrm{d}H_i^{(k)} - \sum_r Z_r\big(H_i^{(k)}\big)\,\mathrm{d}H_0^{(r)}\right)\\ &= \Omega_0 X_{i+1}^{(k)} - \sum_r Z_r\big(H_i^{(k)}\big)\Omega_0 X_1^{(r)}\end{aligned}$$

and the first bracket. Additionally, Poisson tensors Π_0 and Π_{1d} are compatible as

$$\begin{aligned}[\Pi_{1d}, \Pi_0]_S &= \left[\Pi_1 - \sum_i X_1^{(i)} \wedge Z_i, \Pi_0\right]_S \\ &= \sum_i \left(X_1^{(i)} \wedge [Z_i, \Pi_0]_S - Z_i \wedge \left[X_1^{(i)}, \Pi_0\right]_S\right) \\ &= \sum_i \left(X_1^{(i)} \wedge L_{Z_i}\Pi_0 - Z_i \wedge L_{X_1^{(i)}}\Pi_0\right) \\ &= 0.\end{aligned}$$

As a consequence, $(\Pi_0, \overline{\Omega}_\lambda)$ is a Poisson pair and

$$\overline{\Omega}_\lambda(X_i, X_j) = 0.$$

Now, let us consider the presymplectic pencil

$$\Omega_\lambda = \Omega_1 - \lambda\Omega_0.$$

As (Π_0, Ω_1) is a Poisson pair equivalent to the Poisson pair $(\Pi_0, \overline{\Omega}_1)$, then

$$\Omega_\lambda(X_i, X_j) = 0.$$

Moreover, chains (30) take the form

$$\beta_i^{(j)} = \Omega_0 Y_i^{(j)} = \Omega_1 Y_{i-1}^{(j)} - \sum_k B_{i-1,k}^{(j)}\, \mathrm{d}H_0^{(k)}, \qquad B_{i,k}^{(j)} = \sum_l A_{kl} Z_l\left(H_i^{(j)}\right),$$

where $j = 1, \ldots, r$, $i = 0, \ldots, n_j + 1$.

4. Reduction procedure for Gel'fand–Zakharevich chains

Let us consider a $(2n + r)$-dimensional manifold $\mathcal{M}$ and $2n$-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$. Then, let us fix an integrable distribution $\mathcal{Z}$ of constant dimension r that is transversal to $\mathcal{N}$. As mentioned in section 2, such a case is realized by an appropriate dual P–p pair defined on $\mathcal{M}$. Indeed, let (Π_0, Ω_0) be a dual P–p pair on $\mathcal{M}$ with $\ker\Omega_0 = \mathcal{Z} = Sp\{Z_i\}$ and $\ker\Pi_0 = \mathcal{Z}^* = Sp\{\mathrm{d}c_i\}$, $i = 1, \ldots, r$ where obviously $Z_i(c_j) = \delta_{ij}$ and $[Z_i, Z_j] = 0$. Then, $\mathcal{N}$ is a fixed symplectic leaf of Π and $\mathcal{Z}$ consists of vector fields from $\ker\Omega_0$ evaluated on $\mathcal{N}$. An appropriate decomposition of tangent and cotangent bundle of $\mathcal{M}$ is given by (9).

Definition 16. *A function $F : \mathcal{M} \to \mathbb{R}$ is called invariant with respect to distribution $\mathcal{Z}$ if*

$$L_{Z_i}F = Z_i(F) = 0, \qquad \forall Z_i \in \mathcal{Z}.$$

The set of such functions will be denoted by $\mathcal{A}$.

Definition 17. *The Poisson tensor Π is called invariant with respect to the distribution $\mathcal{Z}$ if functions that are invariant along $\mathcal{Z}$ form a Poisson subalgebra with respect to Π, that is*

$$L_{Z_i}\Pi(\mathrm{d}F, \mathrm{d}G) = 0, \qquad Z_i(F) = Z_i(G) = 0. \tag{31}$$

We will denote this subalgebra by $\mathcal{A}(\Pi)$.

Note that Π_0 is obviously $\mathcal{Z}$-invariant as $L_{Z_i}\Pi_0 = 0$, hence $\mathcal{A}(\Pi_0)$ is also a Poisson subalgebra.

Lemma 18. *If Poisson bivector Π is compatible with a presymplectic form Ω_0, then it is invariant with respect to the distribution $\mathcal{Z} = \ker\Omega_0$.*

Proof. Assume $Z_i(F) = Z_i(G) = 0$ for all i. We have to show that condition (31) is fulfilled. But due to theorem 8 it follows that

$$\begin{aligned} L_{Z_l}\Pi(\mathrm{d}F, \mathrm{d}G) &= \big(L_{Z_l}\Pi\big)(\mathrm{d}F, \mathrm{d}G) = \big\langle \mathrm{d}F, \big(L_{Z_l}\Pi\big)\,\mathrm{d}G\big\rangle \\ &= \left\langle \mathrm{d}F, \left(\sum_i [Z_l, X_i] \wedge Z_i - \frac{1}{2}\sum_{i,j} Z_l(c_{ij}) Z_i \wedge Z_j\right) \mathrm{d}G \right\rangle \\ &= \sum_i (Z_i(G)[Z_l, X_i](F) - Z_i(F)[Z_l, X_i](G)) \\ &\quad - \frac{1}{2}\sum_{i,j} Z_l(c_{ij})[Z_j(G)Z_i(F) - Z_j(F)Z_i(G)] \\ &= 0. \end{aligned}$$

□

The invariance of Poisson tensors given in the form (14) was proved for the first time by Vaisman [15].

As a consequence we conclude that an arbitrary Poisson bivector Π, compatible with a dual P–p pair (Π_0, Ω_0), is reducible onto foliation given by Casimirs of Π_0 along the distribution given by $\ker \Omega_0$. Here we propose a simple constructive method of deriving the reduced operator.

Lemma 19. *Let Π be a Poisson bivector compatible with a dual P–p pair (Π_0, Ω_0) and π a reduction of Π onto a symplectic leaf $\mathcal{N}_\nu$ of Π_0 along the transversal distribution $\mathcal{Z} = \ker \Omega_0$. Then, π can be constructed by a restriction of*

$$\Pi_d = \Pi_0\Omega_0\Pi\Omega_0\Pi_0 = \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2}\sum_{i,j} c_{ij} Z_i \wedge Z_j$$

to $\mathcal{N}_\nu$

$$\pi = \Pi_d|_{\mathcal{N}_\nu}. \tag{32}$$

Proof. From the relation (14) and the fact that for $F, G \in \mathcal{A}$

$$\left\langle \mathrm{d}F, \left(-\sum_i X_i \wedge Z_i + \frac{1}{2}\sum_{i,j} c_{ij} Z_i \wedge Z_j\right) \mathrm{d}G \right\rangle = 0,$$

the Poisson operator Π and its deformation Π_d both act in the same way on the set $\mathcal{A}$, so that both can be used to define the same reduced operator π on $\mathcal{N}_\nu$. But as the image of Π_d is tangent to $\mathcal{N}_\nu$, what follows from the fact that $\ker \Pi_0 \subset \ker \Pi_d$, and Π_d is Poisson, then the projection of Π_d onto $\mathcal{N}_\nu$ means simply its restriction to $\mathcal{N}_\nu$. Obviously, if $\ker \Pi_d = \ker \Pi_0$, then (32) means the restriction of Π_d to its symplectic leaf $\mathcal{N}_\nu$. □

Now we pass to the reduction of bi-Hamiltonian chains in Poisson (25) and presymplectic (30) representations onto symplectic foliation of Π_0. Let us denote the projections of Π_0, Π_1 onto $\mathcal{N}$ along $\mathcal{Z}$ by π_0, π_1 and restrictions of $\big(H_1^{(1)}, \dots, H_{n_r}^{(r)}\big)\big|_{\mathcal{N}}$ to $\mathcal{N}$ by $\big(h_1^{(1)}, \dots, h_{n_r}^{(r)}\big)$.

Proposition 20. *The bi-Poisson chain (25), when reduced to $\mathcal{N}$ takes the form*

$$\pi_1\,\mathrm{d}h_i^{(j)} = \pi_0\,\mathrm{d}h_{i+1}^{(j)} - \sum_{k=1}^{r} \alpha_{ki}^{(j)} \pi_0\,\mathrm{d}h_1^{(k)}, \qquad j = 1, \dots, r, \quad i = 1, \dots, n_j, \tag{33}$$

where $\alpha_{ki}^{(j)} = Z_k\big(H_i^{(j)}\big)\big|_{\mathcal{N}}$.

Proof.

$$
\begin{aligned}
\pi_1\,\mathrm{d}h_i^{(j)} &= \Pi_{1d}|_{\mathcal{N}}\,\mathrm{d}H_i^{(j)}\big|_{\mathcal{N}} = \big(\Pi_{1d}\,\mathrm{d}H_i^{(j)}\big)\big|_{\mathcal{N}} \\
&= \big(\Pi_1\,\mathrm{d}H_i^{(j)}\big)\big|_{\mathcal{N}} - \sum_{k=1}^{r}\big(Z_k\big(H_i^{(j)}\big)X_1^{(k)}\big)\big|_{\mathcal{N}} \\
&= \big(\Pi_0\,\mathrm{d}H_{i+1}^{(j)}\big)|_{\mathcal{N}} - \sum_{k=1}^{r}\big(Z_k\big(H_i^{(j)}\big)\Pi_0\,\mathrm{d}H_1^{(k)}\big)\big|_{\mathcal{N}} \\
&= \Pi_0|_{\mathcal{N}}\,\mathrm{d}H_{i+1}^{(j)}\big|_{\mathcal{N}} - \sum_{k=1}^{r} Z_k\big(H_i^{(j)}\big)\big|_{\mathcal{N}}\Pi_0|_{\mathcal{N}}\,\mathrm{d}H_1^{(k)}\big|_{\mathcal{N}} \\
&= \pi_0\,\mathrm{d}h_{i+1}^{(j)} - \sum_{k=1}^{r} Z_k\big(H_i^{(j)}\big)\big|_{\mathcal{N}}\pi_0\,\mathrm{d}h_1^{(k)}.
\end{aligned}
$$

The second and fifth equalities are valid as in coordinates

$$
\big(x^i, H_0^{(j)}\big), \qquad i = 1, \ldots, 2n, \quad j = 1, \ldots, r \tag{34}
$$

on $\mathcal{M}$, the last r rows and columns of Π_0 and Π_{1d} contain zeros only. Obviously we have

$$
\pi_0\big(\mathrm{d}h_i^{(j)}, \mathrm{d}h_k^{(l)}\big) = \pi_1\big(\mathrm{d}h_i^{(j)}, \mathrm{d}h_k^{(l)}\big) = 0,
$$

which follows from the construction of π_0 and π_1. □

Before we pass to the reduction of presymplectic representation (30), observe that restrictions $\Omega_0|_{\mathcal{N}} = \omega_0, \Omega_1|_{\mathcal{N}} = \overline{\Omega}_1|_{\mathcal{N}} = \omega_1$ are closed 2-forms. Moreover, $\pi_0\,\mathrm{d}h_i^{(j)} := K_i^{(j)} = X_i^{(j)}\big|_{\mathcal{N}}$, where $|_{\mathcal{N}}$ means as usually a restriction, as

$$
X_i^{(j)}\big|_{\mathcal{N}} = \big(\Pi_0\,\mathrm{d}H_i^{(j)}\big)\big|_{\mathcal{N}} = \Pi_0|_{\mathcal{N}}\,\mathrm{d}H_i^{(j)}\big|_{\mathcal{N}} = \pi_0\,\mathrm{d}h_i^{(j)}.
$$

Proposition 21. *When reduced to $\mathcal{N}$, the weakly bi-presymplectic chain (25) takes the form*

$$
\omega_1 K_i^{(j)} = \omega_0 K_{i+1}^{(j)} - \sum_k \alpha_{ki}^{(j)}\omega_0 K_1^{(j)}, \qquad j = 1, \ldots, r, \quad i = 1, \ldots, n_j. \tag{35}
$$

Proof.

$$
\begin{aligned}
\omega_1 K_i^{(j)} &= \overline{\Omega}_1|_{\mathcal{N}} X_i^{(j)}\big|_{\mathcal{N}} = \big(\overline{\Omega}_1 X_i^{(j)}\big)\big|_{\mathcal{N}} = \left(\overline{\Omega}_1\left(Y_i^{(j)} - \sum_k Z_k\big(H_i^{(j)}\big)Z_k\right)\right)\Bigg|_{\mathcal{N}} \\
&= \big(\overline{\Omega}_1 Y_i^{(j)}\big)\big|_{\mathcal{N}} - \sum_k \big(Z_k\big(H_i^{(j)}\big)\beta_1^{(j)}\big)\big|_{\mathcal{N}} \\
&= \big(\Omega_0 Y_{i+1}^{(j)}\big)\big|_{\mathcal{N}} - \sum_k \big(Z_k\big(H_i^{(j)}\big)\Omega_0 Y_1^{(k)}\big)\big|_{\mathcal{N}} \\
&= \big(\Omega_0 X_{i+1}^{(j)}\big)\big|_{\mathcal{N}} - \sum_k \big(Z_k\big(H_i^{(j)}\big)\Omega_0 X_1^{(k)}\big)\big|_{\mathcal{N}} \\
&= \Omega_0|_{\mathcal{N}} X_{i+1}^{(j)}\big|_{\mathcal{N}} - \sum_k Z_k\big(H_i^{(j)}\big)\big|_{\mathcal{N}}\Omega_0|_{\mathcal{N}} X_1^{(k)}\big|_{\mathcal{N}} \\
&= \omega_0 K_{i+1}^{(j)} - \sum_k \alpha_{ki}^{(j)}\omega_0 K_1^{(j)}.
\end{aligned}
$$

The second and seventh equalities are valid as in the coordinates (34) vector fields $X_i^{(j)}$ have the last r components equal to zero. □

Note that

$$\omega_1 = \overline{\Omega}_1|_{\mathcal{N}} = \Omega_{1D}|_{\mathcal{N}} = (\Omega_0\Pi_1\Omega_0)|_{\mathcal{N}} = (\Omega_0\Pi_{1d}\Omega_0)|_{\mathcal{N}}$$
$$= \Omega_0|_{\mathcal{N}}\Pi_{1d}|_{\mathcal{N}}\Omega_0|_{\mathcal{N}} = \omega_0\pi_1\omega_0.$$

As ω_1 is closed then (π_1, ω_0) is a compatible pair and $N = \pi_1\omega_0$ is a recursion operator. Moreover $\pi_1 = N\pi_0$ hence π_0 and π_1 are compatible. Now we immediately find that reduced chains (33) and (35) are equivalent. As $K_i^{(j)} = \pi_0\, \mathrm{d}h_i^{(j)}$, hence (35) takes the form

$$N^*\, \mathrm{d}h_i^{(j)} = \mathrm{d}h_{i+1}^{(j)} - \sum_k \alpha_{ki}^{(j)}\, \mathrm{d}h_1^{(j)}, \qquad j = 1, \ldots, r, \quad i = 1, \ldots, n_j, \tag{36}$$

where $N^* = \omega_0\pi_1$ is a recursion operator for 1-forms. On the other hand, multiplying (33) from the left by ω_0 we arrive at (36) again. Moreover,

$$\omega_0\big(K_i^{(j)}, K_l^{(r)}\big) = \pi_0\big(\mathrm{d}h_i^{(j)}, \mathrm{d}h_k^{(l)}\big) = 0, \qquad \omega_1\big(K_i^{(j)}, K_l^{(r)}\big) = \pi_1\big(\mathrm{d}h_i^{(j)}, \mathrm{d}h_k^{(l)}\big) = 0.$$

As a consequence, the distribution tangent to the foliation of $\mathcal{N}$ defined by $\big(h_1^{(1)}, \ldots, h_{n_r}^{(r)}\big)$ is bi-Lagrangian and the n-tuple $\big(h_1^{(1)}, \ldots, h_{n_r}^{(r)}\big)$ of functionally independent Hamiltonians is separable [10]. Separated coordinates are eigenvalues of the recursion operator N and canonically conjugated momenta that put the recursion operator in the diagonal form.

We conclude this section with a statement, that the existence of weakly bi-presymplectic representation of bi-Poisson chains is a sufficient condition for the separability of related Hamiltonian systems.

5. Example

Let us illustrate our previous considerations with a simple nontrivial example of the integrable case of the Henon–Heiles equations

$$(q^1)_{tt} = -3(q^1)^2 - \tfrac{1}{2}(q^2)^2 + c, \qquad (q^2)_{tt} = -q^1q^2. \tag{37}$$

The system (37) can be put into a canonical Hamiltonian form with the Hamiltonian function given by

$$H_1 = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + (q^1)^3 + \tfrac{1}{2}q^1(q^2)^2 - cq^1,$$

where $p_1 = q_t^1$, $p_2 = q_t^2$. The second constant of motion is

$$H_2 = \tfrac{1}{2}q^2p_1p_2 - \tfrac{1}{2}q^1p_2^2 + \tfrac{1}{16}(q^2)^4 + \tfrac{1}{4}(q^1)^2(q^2)^2 - \tfrac{1}{4}c(q^2)^2.$$

The bi-Hamiltonian chain on $\mathcal{M} = Sp(q^1, q^2, p_1, p_2, c)$ is of the following form:

$$\begin{aligned}
\Pi_0\, \mathrm{d}H_0 &= 0\\
\Pi_0\, \mathrm{d}H_1 &= X_1 = \Pi_1\, \mathrm{d}H_0\\
\Pi_0\, \mathrm{d}H_2 &= X_2 = \Pi_1\, \mathrm{d}H_1\\
0 &= \Pi_1\, \mathrm{d}H_2,
\end{aligned}$$

where $H_0 = c$ and the compatible Poisson bivectors are

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & 0 & q^1 & \frac{1}{2}q^2 & p_1 \\ 0 & 0 & \frac{1}{2}q^2 & 0 & p_2 \\ -q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 & -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c \\ -\frac{1}{2}q^2 & 0 & -\frac{1}{2}p_2 & 0 & -q^1q^2 \\ -p_1 & -p_2 & 3(q^1)^2 + \frac{1}{2}(q^2)^2 - c & q^1q^2 & 0 \end{pmatrix}.$$

Now, dual to the canonical Poisson tensor Π_0 is a canonical presymplectic form

$$\Omega_0 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with a kernel vector

$$Z = (0, 0, 0, 0, 1)^T.$$

As evidently $\Omega_0(L_Z\Pi_1)\Omega_0 = 0$, then Π_1 is compatible with the pair (Π_0, Ω_0), so the second presymplectic form is

$$\Omega_1 = \Omega_{1D} + \mathrm{d}H_1 \wedge \mathrm{d}H_0$$
$$= \begin{pmatrix} 0 & -\frac{1}{2}p_2 & -q^1 & -\frac{1}{2}q^2 & 3(q^1)^2 + \frac{1}{2}(q^2)^2 - c \\ \frac{1}{2}p_2 & 0 & -\frac{1}{2}q^2 & 0 & q^1q^2 \\ q^1 & \frac{1}{2}q^2 & 0 & 0 & p_1 \\ \frac{1}{2}q^2 & 0 & 0 & 0 & p_2 \\ -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c & -q^1q^2 & -p_1 & -p_2 & 0 \end{pmatrix}.$$

Hence, the bi-presymplectic representation of the Henon–Heiles chain takes the form

$$\begin{aligned} \Omega_0 Y_0 &= 0 \\ \Omega_0 Y_1 &= \beta_1 = \Omega_1 Y_0 \\ \Omega_0 Y_2 &= \beta_2 = \Omega_1 Y_1 \\ 0 &= \Omega_1 Y_2 \end{aligned}$$

where vector fields Y_i are

$$\begin{aligned} Y_0 &= Z = (0, 0, 0, 0, 1)^T \\ Y_1 &= X_1 + Z(H_1)Z = \left(p_1, p_2, -3(q^1)^2 - \tfrac{1}{2}(q^2)^2 + c, -q^1q^2, -q^1\right)^T \\ Y_2 &= X_1 + Z(H_2)Z = \left(\tfrac{1}{2}q^2p_2, \tfrac{1}{2}q^2p_1 - q^1p_1, \tfrac{1}{2}p_2^2 - \tfrac{1}{2}q^1(q^2)^2,\right. \\ &\quad \left. -\tfrac{1}{2}p_1p_2 - \tfrac{1}{4}(q^2)^3 - \tfrac{1}{2}(q^1)^2q^2 + \tfrac{1}{2}cq^2, -\tfrac{1}{4}(q^2)^2\right)^T. \end{aligned}$$

The chain starts with a kernel vector field Y_0 of Ω_0 and terminates with a kernel vector field Y_2 of Ω_1. The restriction of Π_0, Π_{1d}, Ω_0 and Ω_1 to $\mathcal{N} = Sp(q^1, q^2, p_1, p_2)$ are

$$\pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \omega_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\pi_1 = \begin{pmatrix} 0 & 0 & q^1 & \frac{1}{2}q^2 \\ 0 & 0 & \frac{1}{2}q^2 & 0 \\ -q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 \\ -\frac{1}{2}q^2 & 0 & -\frac{1}{2}p_2 & 0 \end{pmatrix}, \qquad \omega_1 = \begin{pmatrix} 0 & -\frac{1}{2}p_2 & -q^1 & -\frac{1}{2}q^2 \\ \frac{1}{2}p_2 & 0 & -\frac{1}{2}q^2 & 0 \\ q^1 & \frac{1}{2}q^2 & 0 & 0 \\ \frac{1}{2}q^2 & 0 & 0 & 0 \end{pmatrix}$$

with the recursion operator N of the form

$$N = \pi_1\omega_0 = \begin{pmatrix} q^1 & \frac{1}{2}q^2 & 0 & 0 \\ \frac{1}{2}q^2 & 0 & 0 & 0 \\ 0 & \frac{1}{2}p_2 & q^1 & \frac{1}{2}q^2 \\ -\frac{1}{2}p_2 & 0 & \frac{1}{2}q^2 & 0 \end{pmatrix}$$

and $N^* = N^T$.

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